Advanced Probability - Midterm 2018

Attempt all questions. **Time: 2 hours**

All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$.

1. Define the following "distance" function between random variables:

$$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right].$$

Show that $d(X_n, X) \to 0$ if and only if $X_n \xrightarrow{P} X$. **[5 marks]**

2. Let $X_1, X_2, \cdots$ be i.i.d. with $P(X_1 > x) = e^{-x}$. Show that $\limsup_{n\to\infty} X_n/\log n = 1$ a.e. **[5 marks]**

3. Suppose we have a sequence X_n of dependent random variables such that $EX_n = 0$ and $EX_nX_m \leq f(n-m)$ for $m \leq n$ where the function f satisfies $f(k) \to 0$ as $k \to \infty$. Show that $(X_1 + \cdots + X_n)/n \to 0$ in probability. **[5 marks]**

4. Suppose X_n, $n \geq 1$ are independent random variables and $X_n \sim \text{Exp}(\lambda_n)$ (recall that the density is $\lambda_n e^{-\lambda_n x} \cdot \mathbf{1}\{x \geq 0\}$). Show that the series $\sum X_n$ converges almost surely if and only if $\sum_n \frac{1}{\lambda_n}$ is finite. **[7 marks]**

5. Let X_i, $i \geq 1$ be independent random variables with $\lim_{i\to\infty} EX_i = 5$ and $\text{Var}(X_i) = i$. Show that

$$\frac{\sum_{j=1}^n X_j/j}{\sum_{j=1}^n 1/j} \to 5 \quad \text{a.e.} \quad \textbf{[8 marks]}$$